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On the reducibility of the Kampé deFériet function

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Abstract

The existing list of reducible cases of double hypergeometric functions is supplemented by the use of summation theorems for nearly poised hypergeometric functions. A few new relations for single hypergeometric functions emerge as special cases. Relations of this type are useful for computing purposes.

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Various authors have discussed the reducibility of the Kampé de Fériet double hypergeometric function

$$F^{A:}{}_{C:}{}^{B;}{}_{D;}{}^{B'}{}_{D'}\left[\begin{matrix} a_1, \dots, a_A : b_1, \dots, b_B;\ b'_1, \dots, b'_{B'}; \\ c_1, \dots, c_C : d_1, \dots, d_D;\ d'_1, \dots, d'_{D'}; \end{matrix}\ x\ y\right]$$

$$= F^{A:}{}_{C:}{}^{B;}{}_{D;}{}^{B'}{}_{D'}\left[\begin{matrix} (a):(b);\ (b') \\ (c):(d);\ (d'); \end{matrix}\ x, y\right]$$

$$= \sum\ [(a_1, m+n)..(a_A, m+n)(b_1, m)..(b_{B'}, m)(b'_1, n)..(b'_{B'}, n)x^m y^n]/$$

$$[(c_1, m+n)..(c_C, m+n)(d_1, m)..(d_D, m)(d'_1, n)..(d'_{D'}, n)m!n!], \tag{1}$$

where $(a, n) = \Gamma(a+n)/\Gamma(a)$ is the Pochhammer symbol. See, for example [1–7]. Such cases of reducibility are useful in computing these functions.

All indices of summation are taken to run over all of the nonnegative integers and any values of parameters leading to results which do not make sense are tacitly excluded.

The purpose of the present note is to augument the list of such reduction formulae. This is based upon the powerful technique of elementary manipulation of series, combined with the use of summation formulae for nearly poised hypergeometric series. A list of the latter is to be found in

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[6, Appendix III], and is given below for convenience.

$$ {}_3F_2(a, 1 + a/2, \; -n;\, a/2, b;\, 1) = (2 + a - b, n)(b - a - 1, n)/[(b, n)(1 + a - b, n)], \tag{2}$$

$$\begin{aligned} &{}_3F_2(a, b, \; -n;\, 1 + a - b, 1 + 2b - n;\, 1) \\ &\quad = (a - 2b, n)(1 + a/2 - b, n)(-b, n)/[(1 + a - b, n)(a/2 - b, n)(-2b, n)] \end{aligned} \tag{3}$$

and

$$\begin{aligned} &{}_4F_3(a, 1 + a/2, b, \; -n;\, a/2, 1 + a - b, 1 + 2b - n;\, 1) = (a - 2b, n)(-b, n)/ \\ &\quad [(1 + a - b, n)(-2b, n)]. \end{aligned} \tag{4}$$

If (a) denotes the sequence $a_1 a_2 \ldots a_A$, etc., consider the series (supposed to be absolutely convergent)

$$S = [((h), m + p)((a), m)((b), p)x^m y^p]/[((g), m + p)((c), m)((d), p)m!p!], \tag{5}$$

and re-arrange by letting $p = n - m$. We then see that

$$\begin{aligned} S &= [((h), n)((a), m)((b), n - m)x^m y^{n-m}]/[((g), n)((c), m)((d), n - m)m!(n - m)!] \\ &= [(h), n)((b), n)y^n]/[((g), n)((d), n)n!] \\ &\quad \times A + D + 1^F C + B^{((a), 1 - (d) - n, -n; (c), 1 - (b) - n; (-1)^{B - D - 1}x/y)}. \end{aligned} \tag{6}$$

Expression (5) may be interpreted as a Kampé de Fériet function, so that, from (6),

$$\begin{aligned} &F^{H:}{}_{G:}{}^{A;}{}_{C;}{}^{B}{}_{D}\left[\begin{matrix}(h):(a);\ (b) \\ (g):(c);\ (d);\end{matrix}\ x, y\right] \\ &\quad = [((h), n)((b), n)y^n]/[((g), n)((d), n)n!] \\ &\qquad \times A + D + 1^F C + B^{((a), 1 - (d) - n, -n; (c), 1 - (b) - n; (-1)^{B - D - 1}x/y)}. \end{aligned} \tag{7}$$

Reduction formulae then follow by applying (2) to (4), respectively, to sum the inner hypergeometric polynomial on the right-hand side of (7) after appropriate specialisation in each case. Compare Karlsson [4]. These expressions are now listed, noting that two formulae may be deduced from (3).
From (2),

$$\begin{aligned} &F^{H:}{}_{G:}{}^{2;}{}_{2;}{}^{0}{}_{0}\left[\begin{matrix}(h):a, 1 + a/2; - ; \\ (g):a/2,\ b; - ;\end{matrix}\ -y, y\right] \\ &\quad = H + 2^F G + 2^{((h), 2 + a - b, b - a - 1; (g), b, 1 + a - b; y)}, \end{aligned} \tag{8}$$

from (3),

$$\begin{aligned} &F^{H:}{}_{G:}{}^{2;}{}_{1;}{}^{1}{}_{0}\left[\begin{matrix}(h):a, b; -2b; \\ (g):1 + a - b; - ;\end{matrix}\ y, y\right] \\ &\quad = H + 3^F G + 2^{((h), a - 2b, 1 + a/2 - b, -b; (g), 1 + a - b, a/2 - b; y)}, \end{aligned} \tag{9}$$

from (3),

$$F^{H:}{}_{G:}{}^{1;}{}_{0;}{}^{2}{}_{1}\left[\begin{matrix}(h):a;\ a+d-1,-2a; \\ (g):\ -\ ;\ d;\end{matrix}\quad y, y\right]$$

$$= H + 3^F G + 2^{((h), -a, 2-d-2a, d+2a-1; (g), d, 1-d-2a; y)} \tag{10}$$

and from (4),

$$F^{H:}{}_{G:}{}^{3;}{}_{2:}{}^{1}{}_{0}\left[\begin{matrix}(h):a, 1+a/2, b\ ;-2b; \\ (g):a/2, 1+a-b; -\ ;\end{matrix}\quad y, y\right]$$

$$= H + 2^F G + 1^{((h), a-2b, -b; (g), 1+a-b; y)}. \tag{11}$$

Special cases: A few interesting special cases are worthy of note. From (8), let $H = G = 0$, and obtained the formula

$$\exp(y){}_2F_2(a, 1+a/2; a/2, b; -y) = {}_2F_2(2+a-b, b-a-1; b, 1+a-b; y), \tag{12}$$

compare with Kummer's first theorem for confluent hypergeometric functions.

Also, in (9), let $H = 1$ and $G = 0$:

$$(1-y)^{-h}\ {}_3F_2(h, a, 1+a/2; a/2, b; y/(1-y)) = {}_3F_2(h, 2+a-b, b-a-1; b, 1+a-b; y), \tag{13}$$

compare with Euler's transform for the hypergeometric function ${}_2F_1$.

If we let $H = G = 0$ in both (9) and (11), expressions are obtained which are equivalent to a formula given, Eq. (2.4, 2.10), namely,

$${}_2F_1(a, -e/2; 1+a+e/2; x) = (1-x)^e\ {}_3F_2(a+e, 1+a/2+e/2, e/2; 1+a+e/2, a/2+e/2; x). \tag{14}$$

In conclusion, we note that (10) give the result

$$\begin{aligned}{}_2F_1(a+d-1, -2a; d; y) &= (1-y)^{-a}{}_4F_3(a+d-1, -a, 2-d-2a, d \\ &\quad + 2a-1; d, d+a, 1-d-2a; y).\end{aligned} \tag{15}$$

It must be stressed that the above list is by no means exhaustive.

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